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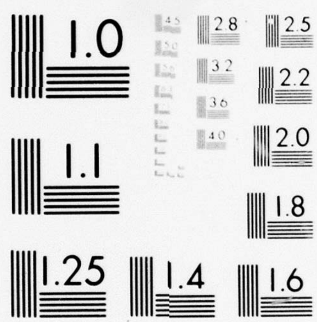
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COMPLETELY INTEGRABLE SYSTEMS AND SYMPLECTIC ACTIONS

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MATHEMATICS RESEARCH CENTER

COMPLETELY INTEGRABLE SYSTEMS AND SYMPLECTIC ACTIONS

M. Adler

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ABSTRACT

We study results on a class of completely integrable systems, for instance with Hamiltonian

$$H(x,y) = \frac{1}{2}\sum_{i=1}^{n} y_i^2 + \sum_{i<j} (x_i - x_j)^{-2} + \alpha \sum_{i=1}^{n} x_i^2 ,$$

using quotient manifolds induced by symplectic group actions, which enables us to integrate the systems and understand their complete integrability. In addition, we give a natural interpretation for the scattering maps associated with these systems.

AMS (MOS) Subject Classification: 34C35

Key Words: Hamiltonian systems, Symplectic group actions, Quotient manifolds, Completely integrable systems, Canonical transformations, Scattering maps

Work Unit Number 1 (Applied Analysis)

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SIGNIFICANCE AND EXPLANATION

We study mechanical problems with high degrees of symmetry, or equivalently many constants of motion. An example of such a system would be n decoupled oscillators, since such a system would have the associated n energies of the oscillators as constants of the motion.

In mechanics, the process of studying systems by ignoring symmetries, or constants of the motion, for instance studying a system in its center of mass coordinates, is well-known. Using a modern abstraction of this old and valuable idea, we study certain systems of interest in mathematical physics, which have many symmetries. We are able to completely solve these systems using the above idea.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

COMPLETELY INTEGRABLE SYSTEMS AND SYMPLECTIC ACTIONS

M. Adler

(1) Introduction - In this short note we explain the results of "Some Finite Dimensional Integrable Systems and Their Scattering Behavior", [1], by the author, and of [5,6,7], in terms of the abstract machinery set up in the paper of D. Kazhdan, B. Kostant, and S. Sternberg, entitled "Hamiltonian Group Actions and Dynamical Systems of Calogero Type", [2], which explains systems first discovered by F. Calogero, C. Marchioro, [3], and first discussed by J. Moser [4].

Briefly, the systems to be discussed have the property that their equations of motion can be expressed as matrix differential equations which can be easily integrated, and moreover, the integration process is seen to occur naturally in a space of much higher dimensionality than the systems in question. The systems to be studied are thus interpreted as quotient systems, of the much larger systems, where the quotienting out process is performed by a symplectic action of the unitary group.

The process of quotienting out in mechanics, such as using center of mass coordinates, i.e. ignoring the position of the center of mass, is indeed a common practice. We point out that usually quotienting out, or ignoring certain data, is a way of ignoring the symmetries, or integrals of the system, so as to arrive at some basic equations to study. Here the quotienting out does not really involve the integrals, but enables us to pass to the ultimate system to be studied. The integrals are in fact generated in a much more trivial way, through the use of natural Lagrangian submanifolds and simple canonical maps, which of course makes use of the quotient structure. In addition, the so-called scattering maps of these systems have a natural interpretation in this context.

In the first section we merely summarize the abstract machinery of [2] of use in the discussion, referring the reader to [2] and the paper of J. Marsden and A. Weinstein [8], for a fuller discussion. We then discuss the results of [1], which entails referring to [1] frequently.

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(2) The Symplectic Structures - We summarize and briefly discuss the necessary abstract machinery needed to discuss [1]. Let (M,ω,G) be a triple, with M an (exact) symplectic manifold with nondegenerate closed two-form $\omega = d\tau$, and G a Lie group, with elements g, which acts on M with a symplectic action. If $\mathcal{L}$ is the Lie algebra of G, with elements denoted by $\dot{g}$, then the action of G associates with each $\dot{g}$ the Hamiltonian vector-field $\hat{\dot{g}}$, and the Hamiltonian function $f_{\dot{g}}(\ ) = -\tau(\hat{\dot{g}})(\ )$, yields a Lie homomorphism, i.e.,

$$\{f_{\dot{g}_1}, f_{\dot{g}_2}\} = f_{[\dot{g}_1,\dot{g}_2]}, \quad [\ ,\ ] \text{ the bracket in } \mathcal{L}, \tag{2.1}$$

where $\{\ ,\ \}$ is just the usual Poisson bracket, i.e. if

$$X_f \lrcorner\, \omega = df, \quad \text{then} \quad X_{f_2}(f_1) = \omega(X_{f_1}, X_{f_2}) \equiv \{f_1, f_2\}. \tag{2.2}$$

We define the moment map of Souriac,

$$\Phi : M \to \mathcal{L}^*, \quad \text{by} \quad \Phi(m)(\dot{g}) = f_{\dot{g}}(m), \tag{2.3}$$

with $\mathcal{L}^*$ the dual of $\mathcal{L}$. The group G acts on itself by conjugation, hence on $\mathcal{L}$ by the linearization of conjugation, Adj, and on $\mathcal{L}^*$ by $(\mathrm{Adj})^*$, and its easy to see that (2.1) is just the infinitesmal, and hence equivalent version of the relation of equivariance,

$$\Phi \circ g = (\mathrm{Adj}\ g^{-1})^* \circ \Phi. \tag{2.4}$$

We then form the orbit of $\alpha \in \mathcal{L}^*$ under $(\mathrm{Adj})^*$, θ_α, and assume $V = \Phi^{-1}(\theta_\alpha)$ is a manifold. Then in fact it is a coisotropic manifold, i.e. $(TV_x)^\perp \subset (TV_x)$, for all $x \in V$, with $\perp$ denoting perpendicularity with respect to ω, and we can thus form $S = V/(\text{leaves of the foilation induced by } (TV)^\perp)$, taking S to be connected and assuming it to be a manifold. Then as a direct consequence of (2.4), it's not hard to see (although it's not shown in [2,8]), that S is a covering space of θ'_α as follows:

$$S \cong \theta'_\alpha \times \theta_\alpha, \quad \theta'_\alpha = \frac{\Phi^{-1}(\alpha)}{G_\alpha}, \tag{2.5}$$

where $/G_\alpha$ means we identify elements $x,y \in M$ if they lie on the same G_α orbit,

with G_α the isotopy group of α, i.e. the connected subgroup of G which fixes α by its action on M. Incidentally, this shows S is a manifold precisely if θ'_α is one. By the coisotropy of V, and the transitivity of G on the fibers θ_α, ω induces a symplectic structure on θ'_α, ω_α, (i.e. we shall tacitly assume ω_α is nondegenerate), where $\{\ ,\ \} \mapsto \{\ ,\ \}_\alpha$ is a homomorphism. The structure $(\theta'_\alpha, \omega_\alpha), G$ shall be our arena of activity.

We note that by (2.1), (2.2), functions in M which are G invariant, induce Hamiltonian flows on M with pointwise fix the image of M under ϕ, and hence they induce Hamiltonian flows on θ'_α. In addition, such functions, if they are in involution with respect to $\{\ ,\ \}$ on M, are via the homomorphism $\omega \to \omega_\alpha$, automatically in involution in θ'_α, thought of, by their G invariance, as functions on θ'_α. This ends our discussion of quotient structures.

In preparation we discuss the M's which shall come up in the examples.

Let F be the linear manifold of $n \times n$ matrices with complex coefficients, and $T^*F = \Phi$ be the cotangent bundle of F, where we shall identify $T^*F \sim F \times F$ via the bilinear form $\langle X,Y \rangle = \operatorname{tr} XY$. Then the complex symplectic 2-form ω, naturally associated with T^*F is

$$W = \Sigma\, dX_{ij} \wedge dY_{ji} = \langle dX, dY \rangle ,$$

or alternately, we write Hamilton's equations, with Hamiltonian $H = H(X,Y)$, as

(2.6) $$\dot{X} = H_Y, \quad \dot{Y} = -H_X ,$$

where $[H_X]_{ij} = \dfrac{\partial H}{\partial X_{ji}}$, etc. If we restrict ω to

(2.7) $$M_1 = T^*\mathcal{L} = \{(X,Y) \mid X = X^*,\ Y = Y^*\} ,$$

where $*$ denotes taking the Hermitian adjoint, i.e. $\mathcal{L}$ is just the self-adjoint matrices, (which we shall identify with the Lie algebra of the unitary group $G = U(n,C)$), ω yields a real symplectic structure, with Hamilton's equations remaining as given in (2.6), where it is understood that H is real.

It is interesting to map $\Phi \mapsto \Phi$ via

(2.8) $$\tau : \begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} X \\ XY \end{pmatrix} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} ,$$

then as one computes

$$\left\{\begin{array}{ll} H_X = H_{Z_1} + YH_{Z_2}, & H_Y = H_{Z_2}X \\ \dot{Z}_1 = \dot{X} = H_Y, & \dot{Z}_2 = (XY)^{\cdot} = H_Y Y - XH_X, \end{array}\right\}$$

and so we may write Hamilton's equations in (Z_1,Z_2) coordinates as

$$\dot{Z}_1 = H_{Z_2}Z_1, \quad \dot{Z}_2 = [H_{Z_2},Z_2] - Z_1H_{Z_1}. \tag{2.9}$$

Let us now restrict τ to $T^*U(n)$, i.e. we identify

$$M_2 = T^*U(n,C) \cong \{(Z_1,Z_2)\,|\,Z_1Z_1^* = I,\ Z_2^* = Z_2\}, \tag{2.10}$$

and restrict $\tau \to \tau_{|\tau^{-1}(M_2)} = \bar{\tau}$. Note $\bar{\tau}$ is invertible, and of course we may just as well identify $\tau^{-1}(M_2)$ with $T^*U(n)$. In that case, since by the pairing $\langle\ ,\ \rangle$, X, Y are the usual dual coordination of $T^*U(n)$, (2.9) restricted to M_2 precisely yields Hamilton's equations for the natural symplectic structure of $T^*U(n)$; that is, up to the factor i which must be put in due to our identification of the Lie algebra of U(n) with self-adjoint matrices, or equivalently we may think of time as being purely imaginary in (2.9). We omit the necessary, but easy verification that (2.9) restricted to M_2 automatically preserves M_2, which is sufficient to insure the restricted ω is symplectic.

(3) Equations of Motion for the Simplest System - We now apply the discussion in Section 2. We shall let (M,ω,G) of Section 1 be $(M_1,(dX,dY),U(n))$ of Section 1. Since $M_1 = T^*\mathcal{L} = \mathcal{L} \times \mathcal{L}$, $\mathcal{L}$ the Lie algebra of $U(n)$, $U(n)$ acts naturally on $\mathcal{L}$ via Adj, i.e. by conjugation which naturally extends to a Hamiltonian action on $T^*\mathcal{L}$ by (if $U \in U(n,c)$)

$$U : (X,Y) \mapsto (UXU^{-1}, UYU^{-1}) ,$$

of which the linear version is

$$\dot{U} : (X,Y) \mapsto ([\dot{U},X],[\dot{U},Y]) \in TM_{1(X,Y)} ,$$

hence, by (2.3), and the above,

$$\Phi(X,Y)(\dot{U}) = f_{\dot{U}}(X,Y) = \langle [\dot{U},X],Y\rangle = \langle [X,Y],\dot{U}\rangle ,$$

and so by the identification of $\mathcal{L}$ with $\mathcal{L}^*$ through $\langle\ ,\ \rangle$,

$$\Delta_\alpha \equiv \Phi^{-1}(\alpha) = \{(X,Y) \mid [X,Y] = \alpha\} , \tag{3.1}$$

and we shall once and for all pick α such that

$$[\alpha]_{jk} = i(1-\delta_{jk}), \quad \text{i.e.} \quad \alpha = i\{v^* \otimes v\} ,$$

with $v \in R^n$, $v = (1,1,1,\ldots,1)^T$. Note that the isotropy subgroup $G_\alpha = \{U \mid U\alpha U^{-1} = \alpha\}$, and we shall define the reduced subgroup

$$G_0 = \{U \mid U(v) = v\}, \quad \text{with Lie algebra} \quad \mathcal{L}_0 = \{B \mid B(v) = 0\} . \tag{3.2}$$

In [2] it is shown by a simple linear algebra argument that if $[X,Y] = \alpha$, we can always find a unique $U \in G_0$ such that

$$\left\{\begin{array}{l} UXU^{-1} = \mathrm{diag}(x_1,x_2,\ldots,x_n) \equiv \bar{x}, \quad x_i < x_{i+1}, \quad \text{all } i \\ [UYU^{-1}]_{jk} = \delta_{jk}y_j + i(1-\delta_{jk})(x_j - x_k)^{-1} \equiv \bar{y} \end{array}\right\} , \tag{3.3}$$

and hence $\theta'_\alpha = \Phi^{-1}(\alpha)/G_\alpha$ is effectively coordinized by $((x_1,x_2,\ldots,x_n),(y_1,\ldots,y_n)) = (x,y)$, and moreover it is shown in [2], by a local argument, that

$$\omega \mapsto \omega_\alpha = \sum_{i=1}^{n} dx_i \wedge dy_i , \quad \text{i.e.} \quad (x,y) \tag{3.4}$$

form a set of canonical coordinates. Hence in this case, θ'_α, and thus S is a manifold (see Section 2), and ω_α is nondegenerate, and hence symplectic.

We now wish to find functions on θ'_α, and by the discussion in Section 2, functions of the form

$$H = H(X,Y) = \operatorname{tr} P(X,Y) \,, \tag{3.5}$$

with $P(\cdot,\cdot)$ a noncommuting polynomial in its arguments will certainly do. If we take (3.5) as a Hamiltonian function on M_1, then (2.6) yields for Hamilton's equations,

$$\dot{X} = h_1(X,Y), \quad \dot{Y} = h_2(X,Y) \,, \tag{3.6}$$

with $h_i(\cdot,\cdot)$, $i = 1,2$, polynomials in their arguments, uniquely determined by $P(\cdot,\cdot)$. As mentioned, $H = \operatorname{tr} P(X,Y)$ automatically can be thought of as a function on θ'_α, in fact via

$$h(X,Y) = \operatorname{Tr} P(\bar{x},\bar{y}), \quad \text{(see (3.11))} \,, \tag{3.7}$$

and we wish to determine the analog of (3.6) for the system on θ'_α with Hamiltonian (3.7), or to put it another way, we shall determine how (3.6) transforms in θ'_α.

So assume we are given initial data $(X(0),Y(0))$ for Hamilton's equations with Hamiltonian $h(x,y)$ in θ'_α, which corresponds to $(\bar{x}(0),\bar{y}(0))$ which we may identify, and thus set equal to, $(X(0),Y(0))$ in M_1. Under the Hamiltonian $h(x,y)$, $(\bar{x}(0),\bar{y}(0)) \mapsto (\bar{x}(t),\bar{y}(t))$, and correspondingly under the Hamiltonian $H(X,Y)$, $(X(0),Y(0)) \mapsto (X(t),Y(t))$. By the previous remarks, we must have

$$X(t) = U\bar{x}(t)U^{-1}, \quad Y(t) = U\bar{y}(t)U^{-1}, \quad U = U(t) \,, \tag{3.8}$$

with $U(t) \in G_0$, (see (3.2)), uniquely defined, as the $H(X,Y)$ flow in the big manifold M_1 descends to the $h(x,y)$ flow in the little manifold θ'_α through quotienting out via G_0. Define $B(t) \in \mathcal{L}_0$, (see (3.2)), by $\dot{U} = -UB$, and so by (3.6), (3.8)

$$\dot{X} = U\delta\bar{x}U^{-1} = h_1(X,Y) = Uh_1(\bar{x},\bar{y})U^{-1} \,,$$

where $\delta\bar{x} = \dot{\bar{x}} - [B,\bar{x}]$, and so we have

$$\delta\bar{x} = h_1(\bar{x},\bar{y}), \quad \delta\bar{y} = h_2(\bar{x},\bar{y}) \,, \tag{3.9}$$

as a consequence of Hamilton's equations on θ'_α, $\dot{x} = \frac{\partial}{\partial x} h(x,y)$, $\dot{y} = -\frac{\partial}{\partial y} h(x,y)$, and thus we see how (3.6) is transformed in θ'_α. Note that from the definition of δ, (3.9), and $B(v) = 0$, (see (3.2)), we can immediately compute the unexpected functional dependence, [illegible] $= B(\bar{x}(t),\bar{y}(t))$, since $\bar{x}$ is a diagonal matrix.

We specialize to the case $H = H_f = \mathrm{tr}\, f(Y)$, for which we compute, (see (3.6)), $h_1(X,Y) = f'(Y)$, $h_2 = 0$, and thus conclude from (3.6), (3.9)

$$\dot{X} = f'(Y), \quad Y' = 0 , \tag{3.10}$$

$$\delta\bar{x} = f'(\bar{y}), \quad \delta\bar{y} = 0 . \tag{3.11}$$

Since (3.10) is immediately solvable, we have in fact solved (3.11) by the use of (3.8).

We also note that since clearly the H_f's are in involution on M_1, being functions only of Y, that by the homomorphism $\omega \mapsto \omega_\alpha$, the h_f's, $h_f = \mathrm{tr}\, f(\bar{y})$, are in involution on θ'_α, and in fact are generated by n independent functions, $h^{(j)} = \mathrm{tr}(\bar{y})^j$, $j = 1,2,\ldots,n$. Thus $h = \mathrm{tr}(\frac{1}{2}\bar{y}^2)$ gives rise to a completely integrable Hamiltonian system.

(4) Scattering Maps - Upon inspection, one observes that the map

(4.1) $$N : (X,Y) \mapsto (Y,X), \quad \eta_{|_{\Delta_\alpha}} : \Delta_\alpha \mapsto \Delta_{-\alpha}, \quad \text{(see (3.1)} ,$$

is a canonical map with multiplier -1 in M_1, and hence so is its projection η_p in θ'_α, since $\omega \mapsto \omega_\alpha$ is a homomorphism. This map, η_p, is precisely the scattering map for system (3.11), with $f(s) = \frac{1}{2} s^2$, which is discussed in [1], Theorem 6, and was observed by J. Moser. More precisely, in the above case, (3.11) with $f(s) = \frac{1}{2} s^2$, one shows the time evolution of the system is given by

$$(x,y) = (qt + p + O(t^{-1}), p + O(t^{-2})), \quad t \to +\infty ,$$

with

$$\eta_p : (x(0),y(0)) \mapsto (q,p) .$$

Similarly one defines the $\frac{\pi}{4}$ rotation map

(4.2) $$\hat{\eta} : (X,Y) \mapsto 2^{-\frac{1}{2}}(X + Y, X - Y), \quad \hat{\eta}_{|_{\Delta_\alpha}} : \Delta_\alpha \mapsto \Delta_\alpha ,$$

which is easily seen to be canonical with multiplier -1 *on* M_1, *(here one uses that* X, Y are Hermitian), and the corresponding canonical projection, $\hat{\eta}_p$ on θ'_α. In fact, in (3.7), if $h(x,y) = \frac{1}{2} \operatorname{tr}(\bar{y}^2 - \bar{x}^2)$, then the time evolution of the system given by (3.9) is given by

$$(x(t),y(t)) = 2^{-\frac{1}{2}} \cdot (q^{\pm}e^{\pm t} + p^{\pm}e^{\mp t}, q^{\pm}e^{\pm t} - pe^{\mp t}) + O(e^{-2|t|}), \quad t \to \pm\infty$$

with

$$\eta_p(x(0),y(x)) = (q,p) .$$

This is shown in [1], Theorem 4. Moreover, as one easily checks

$$\eta = \hat{\eta} \circ \rho \circ \hat{\eta}^{-1}, \quad \rho : (X,Y) = (X,-Y) ,$$

hence

$$\eta_p = \hat{\eta}_p \circ \rho_p \circ \hat{\eta}_p^{-1}, \quad \text{and by (3.3)}, \quad \rho_p(x,y) = (x,-y) .$$

Note that we have shown, by the time reversibility of system (3.9), that $\eta_p : (q^-,p^-) \mapsto (q^+,p^+)$. These latter statements are shown in [1], Theorems 5, 6.

(5) Two Equivalent Systems - We now investigate the systems on θ_α' with Hamiltonians respectively

(5.1) a) $h_1(x,y) = \mathrm{tr}\ f(\bar{x}\cdot\bar{y})$, b) $h_2(x,y) = \mathrm{tr}\ f(\frac{1}{2}[\bar{x}+\bar{y}]\cdot[\bar{x}-\bar{y}])$.

By the canonical map $\hat{\eta}_p$ of Section 4, it is only necessary to investigate case a), and then via $\hat{\eta}_p$, transpose the results to case b). Also, via the transformation formalism (3.8), and (3.6) $\mapsto$ (3.9), it is only necessary to study the equations on the full manifold M_1, rather than the quotient manifold θ_α'. The formalism says substitute δ for $\frac{d}{dt}$, $(\bar{x},\bar{y})$ for (X,Y), which enables one to compute the generator B, and solve the equations on the big manifold, M_1, and via (3.8), pass to the solution on the quotient manifold θ_α'.

We thus need only study the system on M_1 with Hamiltonian

(5.2) $$H_1 = H_f(X,Y) = \mathrm{tr}\ f(XY) ,$$

and we first show the H_f's are in involution. Specifically assume

$$H^{(1)} = \mathrm{tr}\ f_1(XY), \quad H_2 = \mathrm{tr}\ f_2(XY) ,$$

then since, taking increments,

$$\delta H^{(1)} = \mathrm{tr}(f_1'(XY)\cdot[\delta X\cdot Y + X\cdot\delta Y]) + O^{(2)} ,$$

where $O^{(2)}$ means terms of at least second order, we have

(5.3) $$H_X^{(1)} = Y\cdot f_1'(XY), \quad H_Y^{(1)} = f_1'(XY)\cdot X ,$$

etc. for $H^{(2)}$. By (2.6), (2.2), the Poisson bracket $\{\ ,\ \}$ of $H^{(1)}, H^{(2)}$ is given by, (where $(A,B) = \mathrm{tr}(AB)$), $\{H^{(1)},H^{(2)}\} = (H_X^{(1)},H_Y^{(2)}) - (H_Y^{(1)},H_X^{(2)})$, and thus substituting in (5.3), we find

$$\{H^{(1)},H^{(2)}\} = (Yf_1',f_2'X) - (f_1'X,Yf_2') = \mathrm{tr}(f_1'f_2'XY) - \mathrm{tr}(f_1'f_2'XY) = 0 ,$$

where we have used $[f_2'(XY),XY] = 0$, and so $\{H^{(1)},H^{(2)}\} = 0$.

We have thus through $\hat{\eta}$, shown the H_2's, $H_2 = H_f = \mathrm{tr}\ f(\frac{1}{2}[X+Y]\cdot[X-Y])$, are in involution, and thus by the homomorphism $\omega \mapsto \omega_\alpha$, so are the $[\mathrm{tr}\ f(\bar{x}\bar{y})]$'s, $[\mathrm{tr}\ f(\frac{1}{2}[\bar{x}+\bar{y}]\cdot[\bar{x}-\bar{y}])]$'s respectively, and so

$\mathrm{tr}(\frac{1}{2}\,\bar{x}\bar{y})$, $\mathrm{tr}(\frac{1}{2}\,[\bar{x}+\bar{y}]\cdot[\bar{x}-\bar{y}])$ respectively give rise to completely integrable Hamiltonian systems on θ'_α, as observed and proven in [1]. This is thus the second, and more pleasant proof of that fact.

Now by (2.6), (5.3), Hamiltons equations for the system of (5.2) are

$$\dot{X} = f'(XY)\cdot X, \quad \dot{Y} = -Yf'(XY)\ , \tag{5.4}$$

and since $(XY)^{\cdot} = 0$ as a consequence of $[f'(XY), XY] = 0$, we immediately integrate (5.4) to obtain

$$X = e^{f'(X_0Y_0)t}\cdot X_0, \quad Y = Y_0\cdot e^{-f'(X_0Y_0)t}\ , \tag{5.5}$$

where the subscript 0 denotes evaluation at $t = 0$. Thus by (3.9), the corresponding equations on θ'_α for the systems of (5.1)a are

$$\delta\bar{x} = f'(\bar{x}\bar{y})\cdot\bar{x}, \quad \delta\bar{y} = -\bar{y}f'(\bar{x}\bar{y})\ , \tag{5.6}$$

while (3.8) implies the time evolution of (5.6) is given by

$$\mathrm{diag}(x_1, x_2, \ldots, x_n)(t) = U^{-1}e^{f'(X_0Y_0)t}\cdot X_0U, \quad \text{etc.}\ . \tag{5.7}$$

The canonical map $\hat{\eta}$ (5.4), implies the corresponding equations and time evolution for the system with Hamiltonian $H_2 = \mathrm{tr}\, f(\frac{1}{2}\,[X+Y]\cdot[X-Y])$ are

$$\left\{\begin{array}{l} (X+Y)^{\cdot} = -f'(\frac{1}{2}\,[X+Y]\cdot[X-Y])\cdot[X+Y]\ , \\ (X-Y)^{\cdot} = (X-Y)\cdot f'(\frac{1}{2}\,[X+Y]\cdot[X-Y])\ , \\ \text{with} \qquad \{\frac{1}{2}\,[X+Y]\cdot[X-Y]\}^{\cdot} = 0, \quad \text{and} \\ (X+Y) = [\exp - \{tf'(\frac{1}{2}\,[X_0+Y_0]\cdot[X_0-Y_0])\}]\cdot[X_0+Y_0]\ , \\ (X-Y) = [X_0-Y_0]\cdot[\exp\{tf'[X_0+Y_0]\cdot[X_0-Y_0]\}]\ , \end{array}\right\} \tag{5.8}$$

where the changing of minus signs, $t \to -t$, comes about because $\hat{\eta}$ is canonical with multiplier -1. We now use the same procedure, (3.9), as above to transpose (5.8) to θ'_α, i.e. system (b) of (5.1), thus concluding

$$2\delta\bar{x} = [\bar{x}, f'] - [f', \bar{y}]_+, \quad 2\delta\bar{y} = [\bar{y}, f'] - [f', \bar{x}]_+\ , \tag{5.9}$$

where $[A,B]_+ = AB + BA$, $f' = f'(\frac{1}{2}\,[\bar{x}+\bar{y}]\cdot[\bar{x}-\bar{y}])$. Note the simplicity of (5.8), (5.9),

when $f(s) = s$. We could equally well study the systems gotten by 'stretching' $X \mapsto \mu \cdot X$, in (5.1), (5.2), which tend to have compact behavior and thus give rise to periodic solutions for μ purely imaginary, see [1], in fact for $f(s) = s$, $\mu = \sqrt{-1}$, all solutions are periodic with one and the same nonprimitive period.

Note that our studying the case, $H = \mathrm{tr}\, f(XY)$, $h = \mathrm{tr}\, f(\overline{xy})$, makes it unnecessary to study the (Sutherland) case where our manifold is

$$T^* U(n,C) \cong \{(U,R) \mid UU^* = I,\ R = R^*\},$$

and our Hamiltonian is $H(U,R) = H_f = \mathrm{tr}\, f(R)$, with Hamilton's equations given by (2.9), $(U,R) = (z_1, z_2)$, for after the change $t \to it$, we would get the same formal results as (5.4)-(5.7), including the involution statement, via the map τ, (2.8), where we identify $(X, XY) = (Z_1, Z_2)$ with (U,R). We note that the condition $(X,Y) \in \Delta_\alpha$, namely $[X,Y] = \alpha$ is transformed into $[U, U^{-1}R] = \alpha$, i.e. $R - U^{-1}RU = \alpha$.

(6) Yet Another System on θ_α - We now discuss the system of Section 6 of [1].

One may either regard the Hamiltonian of this system on θ'_α as

(6.1) $$h_1 = \mathrm{tr}(\tfrac{1}{2}(\bar{x}\bar{y})^2 + \bar{x}) ,$$

or

(6.2) $$h_2 = \mathrm{tr}(\tfrac{1}{2}\bar{x}\bar{y}^2\bar{x} + \bar{x}) ,$$

as h_1, h_2 differ by a constant. Of course in the full manifold the corresponding Hamiltonians,

(6.3) $$H_1 = h[\tfrac{1}{2}(XY)^2 + X] ,$$

(6.4) $$H_2 = \mathrm{tr}[\tfrac{1}{2}(XY^2X) + X] ,$$

are far from identical. *Although it is shown in [1] that (6.1,2) is a completely* integrable system, we shall not show (6.3,4) are completely integrable systems, in fact we have not been able to do this.

We shall study both (6.3,4), and then relate them in case the associated differential equations on θ'_α have the same initial data. Since the calculations are so similar to those of Section 5, we just give the results. For simplicity we set $XY = Z$. Then with the Hamiltonian of (6.3) we calculate, from (2.6),

$$\dot{X} = ZX, \quad \dot{Z} = -X ,$$

and since $[X,Y] = \alpha$, $(YX)^{\cdot} = -X$, we have $\frac{1}{2}(XY^2X) + X = e_1$, e_1 a constant, and thus we arrive at, again using $[X,Y] = \alpha$,

(6.5) $$\frac{1}{2}Z^2 - \frac{1}{2}Z\alpha - \dot{Z} = e_1 .$$

Letting $Z = -2a_1^{-1}\dot{a}_1$, we find $\ddot{a}_1 = \frac{1}{2}a_1e_1 - \frac{1}{2}\dot{a}_1\alpha$, hence we have

(6.7) $$\left\{\begin{array}{l} \begin{pmatrix} a_1 \\ \dot{a}_1 \end{pmatrix} = \begin{pmatrix} a_1(0) \\ \dot{a}_1(0) \end{pmatrix} \cdot \exp C_1 t , \\[2ex] C_1 = D_1(X,Y) = \begin{bmatrix} 0 & , & I \\ \frac{1}{4}(XY^2X) + \frac{1}{2}X, & & -\frac{\alpha}{2} \end{bmatrix} \text{ at } t = 0 . \end{array}\right.$$

For the Hamiltonian of (6.4), we find

$$\dot{Z} - [\frac{1}{2} YX, Z] = -X$$

$$\dot{X} - [\frac{1}{2} YX, X] = \frac{1}{2} (XZ + ZX) ,$$

which motivates us to define the derivation δ,

$$\delta(\cdot) = \frac{d}{dt} (\cdot) - [\frac{1}{2} YX, \cdot] ,$$

and thus we have from the above,

$$\delta Z = -X, \quad \delta X = \frac{1}{2} (XZ + ZX) . \tag{6.8}$$

Clearly if $U(0) = I$, $\dot{U} = -U(\frac{1}{2} YX)$, we have the following rule of transformations for matrices $A = A(t)$: if $\hat{A} \equiv UAU^{-1}$, then $\left(\frac{d}{dt}\right)^j \hat{A} = U\delta^j AU^{-1}$. From (6.8) we conclude, using that δ is a derivation,

$$\delta(\frac{1}{2} Z^2 + X) = 0, \quad \delta(\frac{1}{2} Z^2 - \delta Z) = 0 , \tag{6.9}$$

hence by our rule of transformation,

$$\frac{1}{2} \hat{Z}^2 - \dot{\hat{Z}} = e_2, \ e_2 \text{ a constant} ,$$

and thus letting $\hat{Z} = -2a_2^{-1}\dot{a}_2$, we find

$$\ddot{a}_2 = \frac{1}{2} a_2 e_2, \quad \text{i.e.}$$

$$\left\{\begin{array}{l} \begin{pmatrix} a_2 \\ \dot{a}_2 \end{pmatrix} = \begin{pmatrix} a_2(0) \\ \dot{a}_2(0) \end{pmatrix} \cdot \exp C_2 t , \\ C_2 = D_2(X,Y) = \begin{bmatrix} 0 & , \ I \\ \frac{1}{4} (XY)^2 + \frac{1}{2} X, & 0 \end{bmatrix} \text{ at } t = 0 . \end{array}\right\} \tag{6.10}$$

We now consider the case where the $X(0)$ of both systems are the same in θ'_α, and moreover $x_1(0) > 0$. Then since both equations (6.3,4) are the same as seen in θ'_α, they both must have the same long term behavior of X as projected down into θ'_α. In [1], Section 6, it is shown that systems (6.3,4) have the following long term 'scattering' behavior:

$$(\log x,\ xy) = (\lambda^{\pm} t + \beta^{\pm} + O(t^{-1}),\ \pm \lambda + O(t^{-2})),\quad t \to \pm\infty ,$$

where $\log x = (\log x_1, \ldots, \log x_n)$, $xy = (x_1 y_1, \ldots, x_n y_n)$. By arguments in that same section, it's clear that the spectrum of C_1, or alternately C_2, precisely carry the data $\lambda^{\pm}$, and hence we must have

$$D_1(X,Y) \sim D_2(X,Y) ,$$

where $\sim$ denotes spectral equivalence. By (6.7,10), (3.8), this implies

$$D_1(\bar{x},\bar{y}) \sim D_2(\bar{x},\bar{y}) . \tag{6.11}$$

Moreover it follows from (6.5,9) that $D_1(X,Y)$, $D_2(X,Y)$ are isospectral matrices of the differential equations (6.3,4), and so in particular $D_1(\bar{x},\bar{y})$, $D_2(\bar{x},\bar{y})$ are for the (6.1,2) flow, and thus we arrive at

$$D_1(\bar{x}(0),\bar{y}(0)) \sim \lim_{t\to\pm\infty} D_1(\bar{x}(t),\bar{y}(t)) \sim \lim_{t\to\pm\infty} D_2(\bar{x}(t),\bar{y}(t)) , \tag{6.12}$$

which yields the scattering behavior of system (6.1,2) as discussed in Corollaries 11.2, 11.3, of [1], which essentially maintains that the system scatters as if it is completely decoupled, and just constrained to maintain a fixed order on the line.

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We study results on a class of completely integrable systems, for instance with Hamiltonian

$$H(x,y) = \frac{1}{2}\sum_{i=1}^{n} y_i^2 + \sum_{i<j} (x_i - x_j)^{-2} + \alpha \sum_{i=1}^{n} x_i^2 ,$$

using quotient manifolds induced by symplectic group actions, which enables us to integrate the systems and understand their complete integrability. In addition, we give a natural interpretation for the scattering maps associated with these systems.

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